MATHEMATICS

ON THE Δ-NUCLEARITY OF Λ-NUCLEAR FRECHET SPACES

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INTRODUCTION

In this paper we give sufficient conditions for a Λ-nuclear Frechet space to be Δ-nuclear and we apply our results to Λ-nuclear sequence spaces.

The basic material consists of the notion of Λ-type operator between a locally convex space E and a sequence space Λ, introduced by the author in [2], and of the corresponding notion of left Λ-mapping studied in [3].

In § 1 we recall the definition of these operators and we prove some additional properties, needed for this paper.

In § 2 we consider Λ-nuclear locally convex spaces and we obtain an operator characterization for the Λ-nuclearity of a certain class of locally convex spaces, which includes the Frechet spaces. From this we deduce sufficient conditions for a Λ-nuclear space, belonging to this particular class, as well as for its strong dual, to be Δ-nuclear. Finally these results are used to prove some rather simple sufficient conditions for the Δ-nuclearity of nuclear, Λ-nuclear sequence spaces and their strong duals.

All the classical properties, notions and notations concerning locally convex spaces, as well as the elementary theory of sequence spaces will be taken from [5]. They will be used without any further reference. The same will be done as far as the nuclear spaces are concerned. Here we refer to [7].

If not specified, $E, F, \ldots$ will denote complete locally convex Hausdorff spaces.

By an operator we understand a continuous linear mapping. The vector space of all the operators from E to F will be denoted by $L(E, F)$.

If Λ and Δ are sequence spaces we define as usual

$$\Lambda \cdot \Delta = \{(\alpha_n \beta_n) | \alpha = (\alpha_n) \in \Lambda,\ \beta = (\beta_n) \in \Delta\}.$$

All the sequence spaces appearing in this paper are of the following type:

Let $P = \{\alpha^n | n = 1, 2, \ldots\}$ be a countable set of sequences, $\alpha^n = (\alpha_i^n)$ with the following properties:

i) $\alpha_i^n \geqslant 0$ for each i and each n
ii) for each i there exists an n such that $\alpha_i^n \neq 0$

iii) for each n: $\alpha_i^{n+1} \geqslant \alpha_i^n$, $i = 1, 2, \ldots$

iv) for each $\alpha^n \in P$, there exists an $\alpha^m \in P$ and a $\beta = (\beta_i) \in l^1$ such that $\alpha_i^n \leqslant \alpha_i^m \cdot \beta_i$, $i = 1, 2, \ldots$.

Define now the sequence space $\Lambda(P)$ by

$$\Lambda(P) = \{\gamma = (\gamma_i) | p_n(\gamma) = \sum_i |\gamma_i| \alpha_i^n < \infty, \ n = 1, 2, \ldots\}.$$

Then $\Lambda(P)$, equipped with the sequence of semi-norms (p_n) is a nuclear Frechet space. In fact iv) is the Grothendieck-Pietsch criterion for the nuclearity of $\Lambda(P)$.

The topological dual space $(\Lambda(P))'$ of $\Lambda(P)$ coincides with the α-dual space $(\Lambda(P))^*$.

The following fundamental structure theorem on nuclear Frechet spaces $\Lambda(P)$ is proved in [6]:

The topology of $\Lambda(P)$ is the normal topology deduced from the dual pair $(\Lambda(P), \Lambda(P)^*)$ and the strong topology on $\Lambda(P)^*$ is exactly the normal topology deduced from the dual pair $(\Lambda(P)^*, \Lambda(P))$. For reasons of simplicity we'll denote these sequence spaces by $\Lambda, \Delta, \ldots$ and their strong duals by $\Lambda^*, \Delta^*, \ldots$.

The generalized sequence space $\Lambda(E)$ is defined by:

$$\Lambda(E) = \{(x_n) | x_n \in E \text{ and } (p(x_n)) \in \Lambda, \ \forall p \in \mathscr{P}\},$$

where $\mathscr{P}$ denotes the family of continuous semi-norms on E. (see [2] and [10]).

A subset B of $\Lambda(E)$ is said to be bounded if the set

$$\{(p(x_n)) | (x_n) \in B\} \text{ is bounded in } \Lambda.$$

For a normal bounded subset R of Λ and a closed absolutely convex, bounded subset B of E, define:

$$[R, B] = \{(x_n) | (x_n) \in \Lambda(E), \ x_n \in E_B \text{ and } (\|x_n\|_B) \in \Lambda\}.$$

Then, the space E is said to be fundamentally Λ-bounded if the collection of all $[R, B]$ forms a fundamental system of bounded sets in $\Lambda(E)$ (see [10]).

§ 1. Λ-TYPE OPERATORS

Definition: ([2] Ch. 6)

i) An operator $f: E \to \Lambda$ is said to be of Λ-type iff f can be written as $f(x) = (\langle x, a_n \rangle)_n$, with $(a_n) \in \Lambda(E'_\beta)$.

ii) An operator $f: \Lambda \to E$ is said to be of Λ-type iff f can be written as $f((\alpha_n)) = \sum_n \alpha_n x_n$, with $(x_n) \in \Lambda^*(E)$.

PROPOSITION 1:

If E is a Banach space, then every operator from E to Λ (resp. from Λ to E) is an operator of Λ-type.

PROOF:

Since Λ is nuclear, every operator from Λ to E is nuclear, while ([2] Ch. 6, § 5) every nuclear operator from Λ to E is of Λ-type. This proves one part of the proposition.

If now $f \in L(E, \Lambda)$, then ${}^{\tau}f \in L(\Lambda^*, E')$.

So, by the nuclearity of Λ^*, ${}^{\tau}f$ is a nuclear operator, ${}^{\tau\tau}f: E'' \to \Lambda$ is nuclear as well and so is the restriction f of ${}^{\tau\tau}f$ to E.

Now every nuclear operator from E to Λ is a Λ-type operator ([2] Ch. 6, § 5) and the lemma is proved.

REMARK:

Proposition 1 remains valid when Λ is replaced by Λ^*.

LEMMA 1:

i) If E is a quasi-barrelled space such that E'_β is fundamentally Λ^*-bounded, then every Λ^*-type operator $f: E \to \Lambda^*$ can be written as $f(x) = (\lambda_n \langle x, y_n \rangle)$ with $(\lambda_n) \in \Lambda^*$ and where (y_n) is an equicontinuous sequence in E.

ii) If E is reflexive and fundamentally Λ-bounded, then every Λ-type operator $f: E'_\beta \to \Lambda$ can be written as $f(x) = (\lambda_n \langle x, y_n \rangle)$ with $(\lambda_n) \in \Lambda$ and where (y_n) is a bounded sequence in E.

PROOF:

i) The operator f can be written as $f(x) = (\langle x, a_n \rangle)$, where $(a_n) \in \Lambda(E'_\beta)$.

So it is left to prove that (a_n) can be written as $(a_n) = (\lambda_n y_n)$, where (λ_n) and (y_n) are as in i).

Now this is an immediate consequence of the results of ROSIER on the dual space of a generalized sequence space. (See [10], prop. (10) § 7, (7) § 8 and (5) § 9.)

ii) Is proved in the same way.

LEMMA 2:

i) Suppose E is a quasi-barrelled space such that E'_β is fundamentally Λ^*-bounded and let $f: E \to \Lambda^*$ be a Λ^*-type operator. Then there exists a barrelled zero-neighbourhood U in E such that f factorizes as $f = \bar{f} \circ \varphi_U$ where $\bar{f}$ is a Λ^*-type operator from $\hat{E}_U$ to Λ^* and φ_U is the canonical mapping $\varphi_U: E \to \hat{E}_U$.

ii) Suppose E is a reflexive, fundamentally Λ-bounded space and let $f: E'_\beta \to \Lambda$ be a Λ-type operator. Then there exists a barrelled zero-neighbourhood U in E'_β such that f factorizes as $f = \bar{f} \circ \varphi_U$, where $\bar{f}$ is a Λ-type operator from $(E'_\beta)\hat{}_U$ to Λ.

PROOF:

i) By lemma 1, i), f can be written as $f(x) = (\lambda_n \langle x, y_n \rangle)$, with $(\lambda_n) \in \Lambda^*$ and where (y_n) is equicontinuous in E'.

Let M stand for the weakly closed convex hull of the sequence (y_n). Then $M^0 = U$ is a zero-neighbourhood in E.

Since $\varphi_U(x) = 0$ implies $\langle x, y_n \rangle = 0$ for $n = 1, 2, \ldots$, we have a factorization

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\varphi_U"'] & \Lambda^* \\ E_U \arrow[ur, "\psi"'] & \end{tikzcd}$$

where ψ is an operator.

Let $\hat{f}$ stand for the extension of ψ to $\hat{E}_U$.

Now every y_n generates a continuous linear form $\bar{y}_n$ on $\hat{E}_U$ with norm $\|\bar{y}_n\| \leqslant 1$, and obviously, putting $\tilde{x} = \varphi_U(x)$ we obtain:

$$\hat{f}(\tilde{x}) = (\lambda_n \langle \tilde{x}, \bar{y}_n \rangle).$$

Since $(\bar{y}_n)$ is equicontinuous in $(\hat{E}_U)'$, the lemma is proved.

ii) Is proved in the same way.

REMARK:

It is proved in [10] (p. 79) that every Frechet space is fundamentally Λ-bounded and that every DF-space is fundamentally Λ^*-bounded.

Hence lemmas 1 i) and 2 i) are valid whenever E is a Frechet space while lemmas 1 ii) and 2 ii) are valid whenever E is a reflexive Frechet space.

DEFINITION: ([3])

An operator $f: E \to F$ is said to be a left Λ-mapping iff f can be factorized through Λ as $f = \psi \circ \varphi$, where $\varphi: E \to \Lambda$ is an operator of Λ-type and ψ is an operator from Λ to E.

PROPOSITION 2:

If E and F are Banach spaces, then $f: E \to F$ is a left Λ-mapping if and only if f is Λ-nuclear in the sense of [4].

PROOF:

If follows from proposition 1 that a left Λ-mapping $f: E \to F$ can be written as $f(x) = \sum_n \langle x, a_n \rangle\, y_n$, with $(a_n) \in \Lambda(E')$ and $(y_n) \in \Lambda^*(F)$.

Then also $f(x) = \sum_n \|a_n\| \langle x, a_n/\|a_n\| \rangle\, y_n$, with $(\|a_n\|) \in \Lambda$. $(a_n/\|a_n\|)$ is a bounded sequence in E' and where obviously the sequence $(y_n) \subset F$ satisfies the requirement $(\langle y_n, b \rangle) \in \Lambda^*$, $\forall b \in F'$. Hence F is Λ-nuclear.

Suppose on the other hand that f is Λ-nuclear, i.e. that f can be written as $f(x) = \sum_n \lambda_n \langle x, a_n \rangle\, y_n$, with $(\lambda_n) \in \Lambda$, (a_n) bounded in E and

$$(\langle y_n, b \rangle) \in \Lambda^*, \quad \forall b \in F'.$$

Then define

$$\varphi\colon E \to \Lambda \text{ by } \varphi(x) = (\lambda_n\langle x, a_n\rangle)$$

and

$$\psi\colon \Lambda \to F \text{ by } \psi((\alpha_n)) = \textstyle\sum_n \alpha_n y_n.$$

Obviously $f = \psi \circ \varphi$ and φ is a Λ-type operator.

Further, since the series $\sum_n \alpha_n\langle y_n, b\rangle$ is absolutely convergent for all $(\alpha_n) \in \Lambda$ and all $b \in F'$, the series $\sum_n \alpha_n y_n$ is convergent in F for all $(\alpha_n) \in \Lambda$ and the mapping ψ is well defined. Finally ψ is continuous by the Banach-Steinhaus theorem.

COROLLARY 1:

If E and F are Banach spaces, then $f\colon E \to F$ is a left Λ-mapping if and only if it is a left Λ^*-mapping.

This follows immediately from [9] Cor. (2.2).

§ 2. Λ-NUCLEAR SPACES

DEFINITION: ([4] p. 39)

A locally convex space E is said to be Λ-nuclear, if for every barrelled zero-neighbourhood U in E there exists a barrelled zero-neighbourhood $V \subset U$ such that the canonical operator $\varphi_{V,U}\colon \hat{E}_V \to \hat{E}_U$ is a left Λ-mapping.

As an immediate consequence of corollary 1 § 1 we have:

PROPOSITION 1:

E is Λ-nuclear if and only if E is Λ^*-nuclear.

PROPOSITION 2:

If E is Λ-nuclear and F is any Banach space, then every operator from E to F is a left Λ-mapping and a left Λ^*-mapping.

PROOF:

Suppose E is Λ-nuclear and let f be an operator from E to a Banach space F.

Then, if B stands for the unit ball in F and $U = f^{-1}(B)$, we have a canonical factorization

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "\varphi"'] & Im\, f \arrow[r, "i"] & F \\ & \hat{E}_U \arrow[u, "\hat{f}"'] & \end{tikzcd}$$

Now by the Λ-nuclearity of E there is a barrelled zero-neighbourhood V in E such that $\varphi = \varphi_{V,U} \circ \varphi_V$ is a left Λ-mapping.

Thus we obtain a commutative schema:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\varphi_V"'] \arrow[dr, "\varphi"] & Im\, f \arrow[r, "i"] & F \\ \hat{E}_V \arrow[r, "\varphi_{V,U}"'] & \hat{E}_U \arrow[u, "\bar{f}"'] & \end{tikzcd}$$

Hence $f = i \circ \bar{f} \circ \varphi_{V,U}$ is still a left Λ-mapping. ([3] prop. 11).

Since E is also Λ^*-nuclear, it is proved in the same way that f is a left Λ^*-mapping.

PROPOSITION 3:

i) Suppose E is a quasi-barrelled space such that E'_β is fundamentally Λ^*-bounded (in particular E can be a Frechet space). If every operator from E to any Banach space F is a left Λ^*-mapping, then E is Λ-nuclear.

ii) Suppose E is a reflexive, fundamentally Λ-bounded space (in particular E can be a reflexive Frechet space).

If every operator from E'_β to any Banach space F is a left Λ-mapping, then E'_β is Λ-nuclear.

PROOF:

i) Let U be a barrelled zero-neighbourhood in E. Then $\varphi_U: E \to \hat{E}_U$ is a left Λ^*-mapping.

Hence φ_U can be factorized as $\varphi_U = \psi \circ \varphi$ where $\varphi: E \to \Lambda^*$ is an operator of Λ^*-type and $\psi: \Lambda^* \to \hat{E}_U$ is an operator.

Let V stand for the zero-neighbourhood produced in lemma 2 i), § 1 and put $W = U \cap V$.

We then obtain a commutative schema:

$$\begin{tikzcd} E \arrow[r, "\varphi_U"] \arrow[d, "\varphi_W"'] \arrow[dr, "\varphi"] & \hat{E}_U \\ \hat{E}_W \arrow[r, "\bar{\varphi}"'] & \Lambda^* \arrow[u, "\psi"'] \end{tikzcd}$$

where $\bar{\varphi}$ is a Λ^*-type operator.

Hence $\psi \circ \bar{\varphi}$ is a Λ-type operator and i) is proved.

ii) Is proved in the same way, making use of lemma 2, ii), § 1.

PROPOSITION 4:

i) Let E be a quasi-barrelled space such that E'_β is fundamentally Λ^*-bounded (e.g. E is a Frechet space).

Then:

a) If E is Λ-nuclear and Λ is Δ-nuclear, then E is Δ-nuclear.

b) If E is Λ-nuclear and Λ^* is Δ-nuclear, then E is Δ-nuclear.

ii) Let E be a reflexive, fundamentally Λ-bounded space (e.g. E is a reflexive Frechet space).

Then:

a) If E'_β is Λ-nuclear and Λ is Δ-nuclear, then E'_β is Δ-nuclear.

b) If E'_β is Λ-nuclear and Λ^* is Δ-nuclear, then E'_β is Δ-nuclear.

PROOF:

i) Let F be a Banach space and f an operator from E to F. Then by prop. 2, f is a left Λ-mapping and we have $f = \psi \circ \varphi$ where $\varphi: E \to \Lambda$ is an operator of Λ-type. Further, since Λ is Δ-nuclear, the operator $\psi: \Lambda \to F$ is a left Δ^*-mapping (prop. 2).

Thus we obtain a commutative schema:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\varphi"'] & F \\ \Lambda \arrow[ur, "\psi"] \arrow[r, "\chi"'] & \Delta^* \arrow[u, "\varrho"'] \end{tikzcd}$$

where χ is an operator of Δ^*-type.

Hence $\varrho \circ \chi$ is a left Δ^*-mapping and so is f ([3] prop. 11).

By proposition 3 i), E is Δ-nuclear.

The remaining parts of this proposition are proved in the same way.

PROPOSITION 5:

i) If $\Lambda^* \cdot \Lambda \subset \Delta \subset l^1$, then Λ^* is Δ-nuclear.

ii) If $\Lambda^* \cdot \Lambda \subset \Delta^* \subset l^1$, then Λ is Δ-nuclear.

PROOF:

Let F be a Banach space and suppose f is an operator from Λ^* to F.

By proposition 3 it is than sufficient to prove that f is a left Δ-mapping.

By proposition 1 § 1, f can be written as $f((\alpha_n)) = \sum_n \alpha_n x_n$, with $(x_n) \in \Lambda(F)$.

We now consider the linear mappings

$$\varphi: \Lambda^* \to \Delta : (\alpha_n) \to (\alpha_n \|x_n\|)_n = (\langle (\alpha_n), \|x_n\| e_n \rangle)_n$$

and

$$\psi: \Delta \to F : (\beta_n) \to \sum_n \beta_n \, x_n / \|x_n\|$$

(the terms for which $x_n = 0$ are omitted.

Both mappings are well-defined by our assumptions on Λ and Δ.

Further, for every $\gamma = (\gamma_n) \in \Lambda^*$, the sequence

$$(p_\gamma(\|x_n\| e_n))_n = (\|x_n\| \cdot |\gamma_n|)_n \in \Delta .$$

Hence φ is an operator of Δ-type.

The mapping ψ being obviously continuous we conclude that $f = \psi \circ \varphi$ is a left Δ-mapping.

Part ii) is proved by the same argument.

Combining propositions 4 and 5 we obtain:

COROLLARY 2:

If $\Lambda^* \cdot \Lambda \subset \Delta \subset l^1$ or $\Lambda^* \cdot \Lambda \subset \Delta^* \subset l^1$, then

i) Every Λ-nuclear Frechet space is Δ-nuclear.

ii) If E is a reflexive Frechet space such that E'_β is Λ-nuclear, then E'_β is Δ-nuclear.

APPLICATIONS TO D_1 AND D_2-SPACES

DEFINITION: (see [1])

i) $\Lambda = \Lambda(P)$ is said to be a D_1-space if

a) $\alpha^1 = (1, 1, 1, \ldots)$

b) $\forall m, \exists p$ such that $(\alpha^m)^2 \leqslant \alpha^p$.

ii) $\Lambda = \Lambda(P)$ is said to be a D_2-space if

a) $\forall n: \lim_k \alpha_n{}^k = 1$

b) $\forall k, \exists m$ such that $\alpha^k \cdot \alpha^p \leqslant (\alpha^m)^2$ for all p.

The class of D_1- (resp. D_2-) spaces contains the class of the smooth sequence spaces of infinite type (resp. of finite type) (see [11]), which in turn contains the power series spaces of infinite type (resp. of finite type) (see [4] and [8]).

It is proved in [9] (prop. (2.4), (2.5)) that Λ has the property $\Lambda \cdot \Lambda^* = \Lambda$ if and only if Λ is a D_1-space and that Λ has the property $\Lambda \cdot \Lambda^* = \Lambda^*$ if and only if Λ is a D_2-space.

The following results are then an immediate consequence of proposition 5.

COROLLARY 3:

i) If Λ and Δ are D_1-spaces such that $\Lambda \subset \Delta$, then Λ^* is Δ-nuclear.

ii) If Λ is a D_2-space and Δ is a D_1-space such that $\Lambda^* \subset \Delta$, then Λ^* is Δ-nuclear.

iii) If Λ and Δ are D_2-spaces such that $\Delta \subset \Lambda$, then Λ is Δ-nuclear.

iv) If Λ is a D_1-space and Δ is a D_2-space such that $\Lambda \subset \Delta^*$, then Λ is Δ-nuclear.

(Properties i) and ii) are also proved in [9]).

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